Example A2–15 The final cadence—the merging of set classes 4–1 (0123) and 4–21 (0246) within the tritone B♭–E.

There each voice enters in turn, outlining the T_{10} form of 4–21 [B♭,C,D,E]. When the cello finally brings in its B♭, the other voices crash back in. Now the entire space between B♭ and E has been filled in. The chromatic tetrachord 4–1 (0123) and the whole-tone tetrachord 4–21 (0246) are merged in this final sonority. The two principal set classes of the passage thus are developed, progress from one to the other, define a large-scale shift in pitch location, and ultimately merge into a single cadential sonority.

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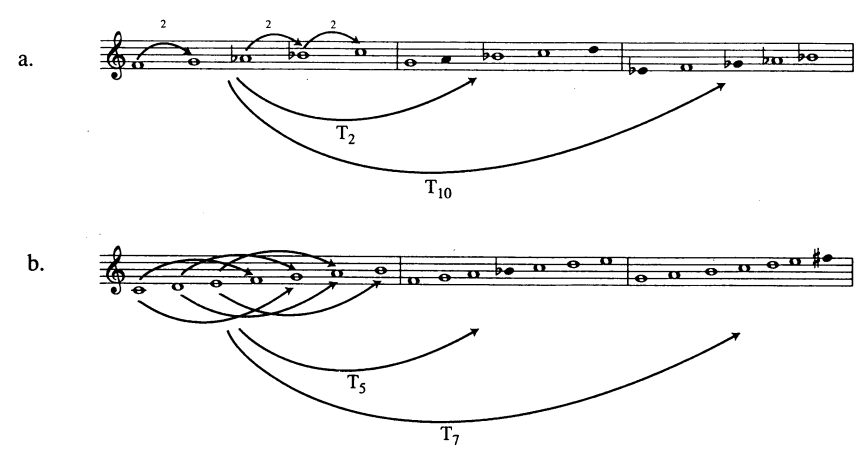
Chapter 3
Some Additional Relationships

Common Tones under Transposition (T_n)

When a pitch-class set is transposed or inverted, its content will change entirely, partially, or not at all. Tones held in common between two different members of the same set class can provide an important musical continuity. Conversely, an absence of common tones may emphasize the contrast between two different members of the same set class.

When you transpose a pitch-class set by interval n, the number of common tones will be equal to the number of times the interval n occurs in the set (with one exception, to be discussed later). If a set contains three occurrences of interval class 2, for example, there will be three common tones at T_2 or T_{10} (see Example 3–1a). The major scale contains six instances of interval class 5, so there will be six common tones when the scale is transposed up or down by five semitones (T_5 or T_7). (See Example 3–1b.)



Example 3–1 Common tones under transposition.

To understand why it works this way, concentrate on the mappings involved. When a set is transposed at T_n, each member of the set maps onto a note that lies n semitones higher. If two of the notes in the set were n semitones apart to begin with, transposing by n semitones maps one of the notes onto the other, producing one common tone. That mapping will happen as many times as there are occurrences of interval n in the set. In other words, for every occurrence of a given interval n, there will be one common tone under T_n.

For example, consider the operation T_3 applied to [4,5,7,8], a member of set class 4–3 (0134). There are two occurrences of interval class 3 in the set, between 4 and 7 and between 5 and 8. As a result, when the set is transposed up three semitones, the 4 maps onto the 7 and the 5 maps onto the 8. Similarly, when it is transposed down three semitones (T_9), the 8 maps onto the 5 and the 7 onto the 4 (see Figure 3–1).

[4, 5, 7, 8] – – – T_3 – – → [7, 8, 10, 11]   [4, 5, 7, 8] – – – T_9 – – → [1, 2, 4, 5]

Figure 3–1

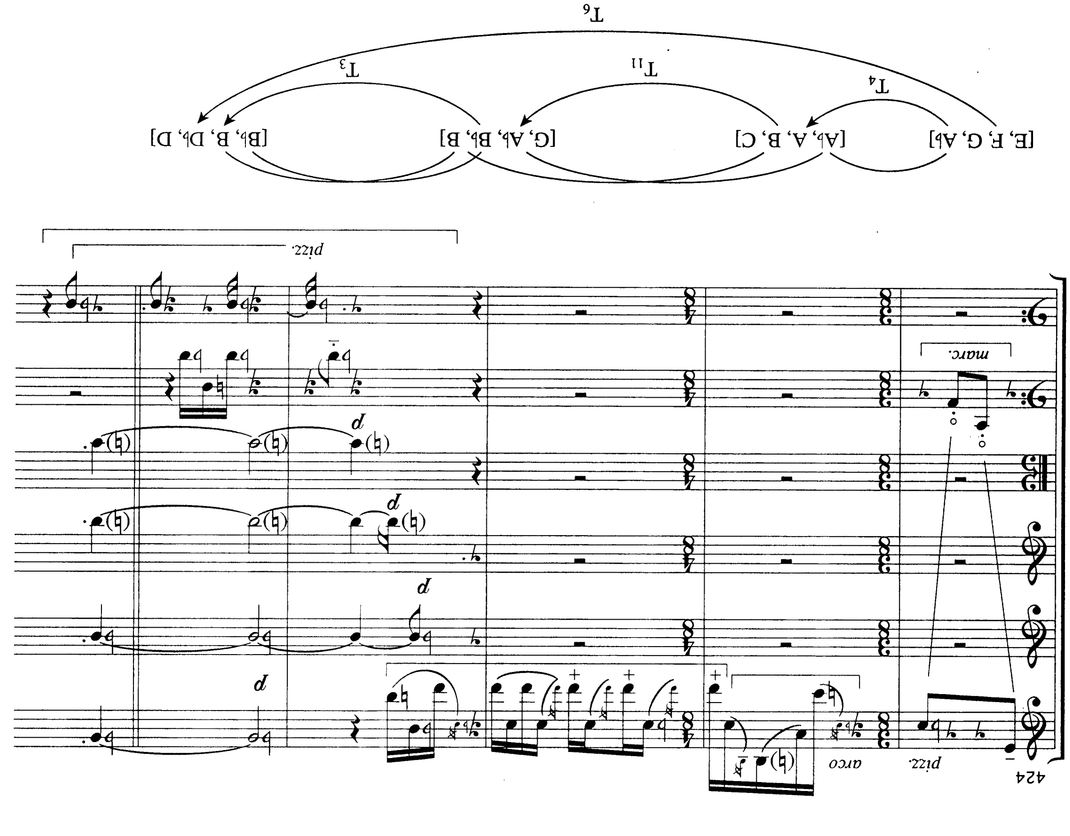
The tritone (interval class 6) is an exception. Because the tritone maps onto itself under transposition at T_6, each occurrence of interval class 6 in a set will create *two* common tones when the set is transposed at T_6. For example, consider [4,9,10], a member of set class 3–5 (016). It contains a single tritone. When the set is transposed at T_6, the 4 maps onto the 10 and the 10 simultaneously maps back onto the 4. As a result, both the 4 and the 10 are held in common at T_6 (see Figure 3–2).

[4, 9, 10] – – T_6 – → [10, 3, 4]

Figure 3–2

To figure out quickly how many common tones a set will have at any transposition level, just look at its interval-class vector. The vector tells you how many times each interval class occurs in any set, which also tells you how many common tones there will be under T_n for any value of n. Set 4–3 (0134), for example, has the vector 212100, and will therefore retain two common tones at T_1 (or T_{11}) and T_3 (or T_9) and a single common tone at T_2 (or T_{10}) and T_4 (or T_8). It will retain no common tones at T_5, T_6, or T_7. These results will hold for all members of the set class. Notice, in Example 3–2, how Stravinsky uses common tones in a passage from *Agon* to create a chain of members of set class 4–3, linked by their common tones. Transposition at T_4 produces one common tone, while transposition at T_{11} and T_3 produces two common tones each. The overall motion, T_6, produces no common tones because the set being transposed contains no i6s.

The interval-class vector for the major scale, set class 7–35 (013568T), is 254361. Notice that it has a different number of occurrences of each interval class. As a result, it will have a different number of common tones at each transpositional level



Example 3–2 A chain of members of set class 4–3 (0134) (Stravinsky, *Agon*).

[1,3,6,9]	
1 + 1 = 2	
3 + 3 = 6	Each of these sums represents
6 + 6 = 0	one common tone.
9 + 9 = 6	

Figure 3–5

For each of these sums, there will be one common tone under T_nI for that value of n.

We can compile all of these sums for [1,3,6,9] into what we will call an *index vector,* remembering that the sums of different elements will yield two common tones, while the sums of elements with themselves will yield one common tone. For each of the twelve possible values of n, we can list the number of common tones under T_nI (see Figure 3–6).

n =	0	1	2	3	4	5	6	7	8	9	10	11
no. of common tones:	3	0	1	2	2	0	2	2	0	2	2	0

Figure 3–6

The largest number of common tones, three, is retained at T_0I, which maps the 3 and the 9 onto each other and the 6 onto itself. Two common tones are retained at T_3I, T_4I, T_6I, T_7I, T_9I, and $T_{10}I$; one common tone is retained at T_2I. No common tones are retained at T_1I, T_5I, T_8I, or $T_{11}I$, because the sums 1, 5, 8, and 11 cannot be produced by adding members of [1,3,6,9] either to each other or to themselves. The number of common tones under T_nI for every value of n can be read from the vector.

A simpler way to figure out the number of common tones under T_nI is to construct an addition table. Write the set along the vertical and horizontal axes and add as indicated. Such an addition table for [3,4,7,8] is shown in Figure 3–7.

	3	4	7	8
3	6	7	10	11
4	7	8	11	0
7	10	11	2	3
8	11	0	3	4

Figure 3–7

This table neatly performs all of the additions required; it adds each element to each other element twice and adds each element to itself once. As a result, each occurrence of a number inside the table represents a single common tone. The number 11 occurs four times, so there will be four common tones at $T_{11}I$; the number 3 occurs twice, so there will be two common tones at T_3I; and so on. It is easy to rearrange this information in the form of an index vector, or simply to read it directly from the table.

This addition table has another advantage—it shows not only how many tones will be held in common under T_nI, but also which ones. Each index number in the table lies at the intersection of two tones. Those are the tones mapped onto one another by that index number. In the table in Figure 3–7, for example, 10 occurs at the intersection of 3 and 7; 3 and 7 are thus held in common at $T_{10}I$. Similarly, 8 occurs in the table at the intersection of 4 with itself, so 4 will be held in common at T_8I.

Appendix 2 lists index vectors for the prime-form of each set class and for the set related by T_0I to the prime form. Unlike the interval vector, the index vector is not the same for every member of a set class. Fortunately, once you know the index vector for the prime form and for its T_0I, the index vectors for all the remaining members can be deduced easily from the simple rules given in Appendix 2. The interval-class vectors in Appendix 1 and the index vectors in Appendix 2 should enable you to find the number of common tones any pitch-class set will retain under T_n or T_nI for any values of n.

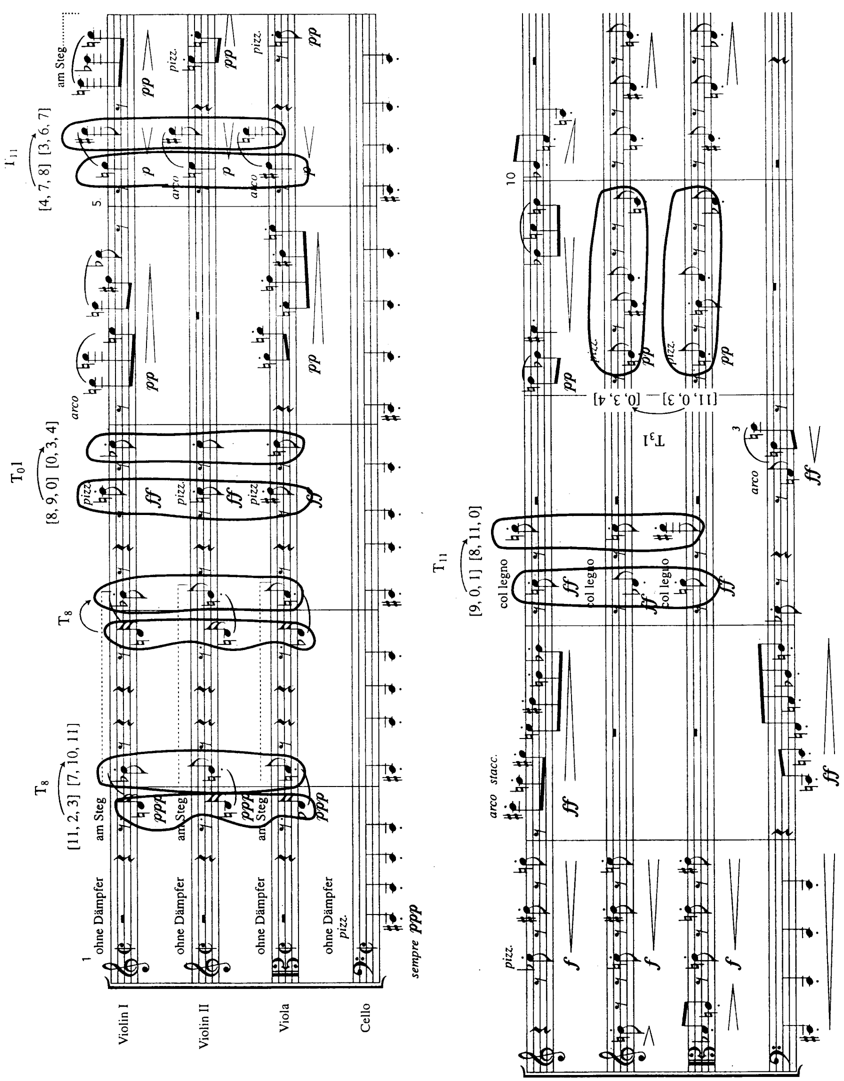
Common tones under T_n and T_nI can be an important source of musical continuity. Example 3–3 contains the first ten measures of the third of Webern's Movements for String Quartet, Op. 5, a composition that makes intensive use of set class 3–3 (014).

Six pairs of 3–3s are marked on the score. Look first at the T_n–related pairs. The transposition levels used, 8 and 11, produce one common tone each, as we know from the interval-class vector of 3–3: 101100. And notice the special treatment this common tone receives in each case—it is always retained in exactly the same register. The common pitch class is expressed as a common pitch.

The same thing is true of the T_nI-related pair in measure 3. The index vector for the first set, [8,9,0], is 100012102200; the index vector for the second set, [0,3,4], is 100220121000. You could discover that either by simply doing an addition table for the sets, as discussed earlier, or by looking up the vectors in Appendix 2 (and performing the proper rotations). Both index vectors show that each of these sets holds one common tone at T_0I. Since they are related by T_0I, that means they will share a single pitch class. That common tone is C, which is retained here not only in the same register but in the same instrument. In measure 9, the two T_3I-related forms of 3–3 share two common tones, C and E♭. Notice how Webern arranges these notes to sound together simultaneously. He thus uses common tones under both T_n and T_nI to create smooth, continuous voice leading as the music progresses among the members of set class 3–3 (014).

Inversional Symmetry

Some set classes contain sets that can map entirely onto themselves under inversion. Such set classes are said to be *inversionally symmetrical* and, of the 220 set classes listed in Appendix 1, seventy-nine have this property. The index vector for a set with this property will have an entry equal to the number of notes in the set.



Example 3–3 Common tones under transposition and inversion (Webern, Movements for String Quartet, Op. 5, No. 3).

Sets that are inversionally symmetrical can be written so that the intervals reading from left to right are the same as the intervals reading from right to left. Usually, but not always, this intervallic palindrome will be apparent when the set is written in normal form. Occasionally, a note has to be written twice to capture the modular wraparound (see Figure 3–8).

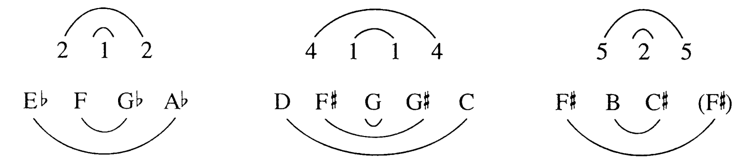


Figure 3–8 Three sets written to display their inversional symmetry.

The sense of inversionally symmetrical sets as their own mirror image is even more apparent when they are written around a pitch-class clockface (see Figure 3–9).

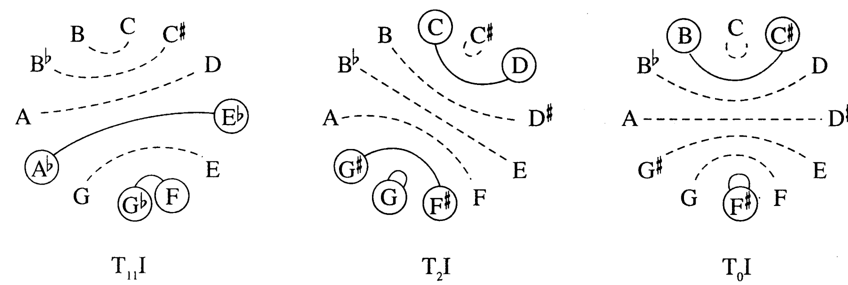
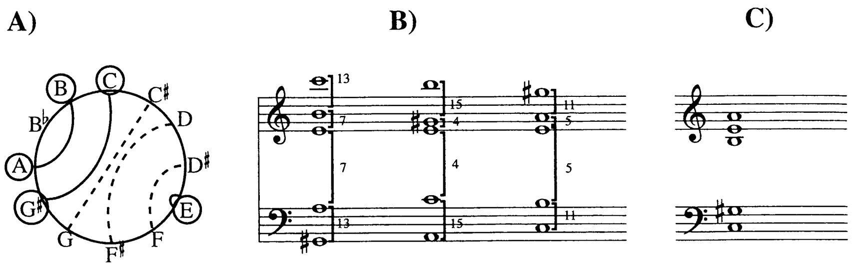


Figure 3–9

In inversionally symmetrical sets, all the notes in the set are mapped either onto other notes in the set or onto themselves under some T_nI. Every note in the set has an inversional partner also in the set.

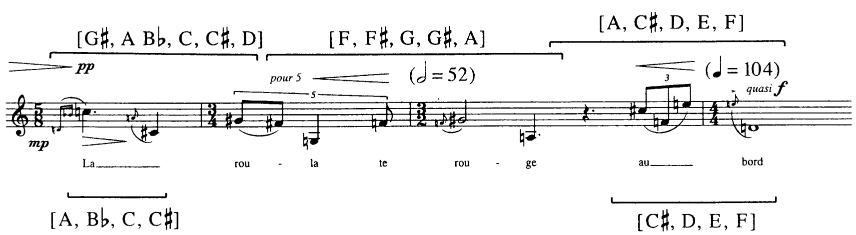
Schoenberg's Orchestra Piece Op. 16, No. 3 begins with the five note chord displayed around a pitch-class clockface in Example 3–4A (the musical score can be found back in Example 2–3). The symmetry of the chord should be apparent—each note of the chord has an inversional partner also within the chord, and the E balances itself as a center of symmetry. It would be easy to write such a chord so as to realize its symmetry in pitch space: the three chords in Example 3–4B arrange those five pitch classes so that they describe the same pitch intervals reading from bottom to top as they do from top to bottom. These three chords are *pitch symmetrical*—they are symmetrical in register and their pitch intervals form a palindrome. Any symmetrical pitch-class set can be arranged symmetrically in pitch space. But that is not what Schoenberg did. His actual chord, shown in Example 3–4C, is *pitch-class*



Example 3–4 Inversional symmetry: A) Schoenberg's chord written on a pitch-class clockface; B) three hypothetical pitch symmetrical arrangements; C) Schoenberg's actual arrangement—symmetry of pitch class, not pitch.

symmetrical, but not pitch symmetrical. The inversional symmetry is still there—every note in the chord has its inversional partner also in the chord—but it is felt only in the more abstract pitch-class space, not in the more concrete, immediate pitch space.

The melody by Boulez in Example 3–5 is based entirely on sets that are inversionally symmetrical.



Example 3–5 Inversionally symmetrical sets (Boulez, *Le Marteau sans Maître,* "L'Artisanat furieux," mm. 6–9, vocal part).

Some of these are arranged pitch-symmetrically, but most are not. Compare, for example, the two forms of sc4–3 (0134): [A, B♭, C, C♯] at the beginning of the melody and [C♯, D, E, F] at the end. The first one is asymmetrical in pitch space (the B♭ would have had to be an octave lower to make it pitch symmetrical), while the second is pitch symmetrical—its pitch intervals are the same reading from bottom to top as from top to bottom.

While inversional symmetry is an important compositional resource generally, it can play a particularly decisive role when the symmetry is realized in pitch space. Any member of set class 3–1 (012), for example, is abstractly symmetrical no matter how it is deployed, but its symmetry can be dramatically reinforced by the arrangement of the notes in register. See, for example, how Varèse arranges the set [C, C♯, D] at the beginning of *Hyperprism* (Example 3–6).

Example 3–6 Pitch symmetry (Varese, *Hyperprism,* mm. 1–12, percussion parts omitted).

C♯ enters in a middle register in the tenor trombone, soon reinforced by horns. At the end of measure 4, D comes in twenty-three semitones below and then, in measure 12, the low D is balanced symmetrically by the high C, 23 semitones above the middle C♯. A vast pitch space is articulated symmetrically, a,nd the symmetry is reinforced by the grace notes in measure 4 (involving the F four semitones above C♯) and the quick embellishment in measure 11 (involving the A four semitones below C♯). In this passage, C♯ is literally the central tone. The role of inversional symmetry in establishing a sense of pitch centricity is a topic to which we will return in Chapter 4.

Like set classes that are transpositionally symmetrical, those that are inversionally symmetrical can be easily identified in Appendix 1. In the middle column, the number after the comma measures the degree of inversional symmetry—it tells the number of inversional levels that map a set onto itself. Many sets cannot map onto themselves under inversion, and thus have a degree of inversional symmetry that is 0. Some sets can map onto themselves at one or more than one inversion levels. The set 3–6 (024), for example, has a degree of symmetry of (1, 1). It maps onto itself at one transpositional level (T_0) and one inversional level (in this case, T_4I). The most symmetrical set of all is the whole-tone scale; it maps onto itself at six transpositional and six inversional levels.

Inversional symmetry is a reasonably common property, but inversional symmetry at more than one level is rare—only the eleven set classes listed in Figure 3–10 have that property.

	Degree of inversional symmetry	
3–12 (048)	3	9–12 (01245689T)
4–9 (0167)	2	8–9 (01236789)
4–25 (0268)	2	8–25 (0124678T)
4–28 (0369)	4	8–28 (0134679T)
6–7 (012678)	2	
6–20 (014589)	3	
6–35 (02468T)	6	

Figure 3–10

It is interesting to compare this list with the list of transpositionally symmetrical sets in Figure 3–3. Virtually all of the set classes that are transpositionally symmetrical are also inversionally symmetrical at more than one level (6–30 is the only exception), and every set class that is inversionally symmetrical at more than one level is also transpositionally symmetrical. And, as noted earlier, set classes with one or both of these properties have often proven attractive to composers, including the augmented triad (3–12), the diminished seventh chord (4–28), and the hexatonic (6–20), whole-tone (6–35), and octatonic (8–28) scales.

The greater the number of operations that map a set onto itself, the smaller the number of distinct sets in the set class. Most set classes have a degree of symmetry of (1,0) and contain twenty-four distinct sets. For all set classes, dividing the number of self-mapping operations into twenty-four will give you the number of sets in the set class. Let's use the prime-form of set 4–9 (0167) as an example to see why this is so. The set-class 4–9 has a degree of symmetry of (2,2). The four operations that map it onto itself are T_0, T_6, T_1I, and T_7I. (This can be figured out by looking at the interval-

class and index vectors.) Now consider another member of this set class, [1,2,7,8]. This is simultaneously T_1, T_7, T_2I, and T_8I of [0,1,6,7]. Each member of the set class can be created by four different operations:

[0,1,6,7]	T_0, T_6, T_1I, T_7I
[1,2,7,8]	T_1, T_7, T_2I, T_8I
[2,3,8,9]	T_2, T_8, T_3I, T_9I
[3,4,9,10]	T_3, T_9, T_4I, $T_{10}I$
[4,5,10,11]	T_4, T_{10}, T_5I, $T_{11}I$
[5,6,11,0]	T_5, T_{11}, T_6I, T_0I

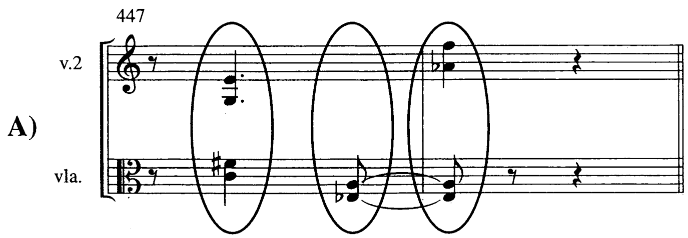
But there are only twenty-four possible operations in all—twelve values of n for T_n, and twelve values of n for T_nI. As a result, twenty-four divided by the number of operations that will produce each member of the set class equals the number of distinct members of the set class. In this case, twenty-four divided by four equals six, and set-class 4–9 has only six members.

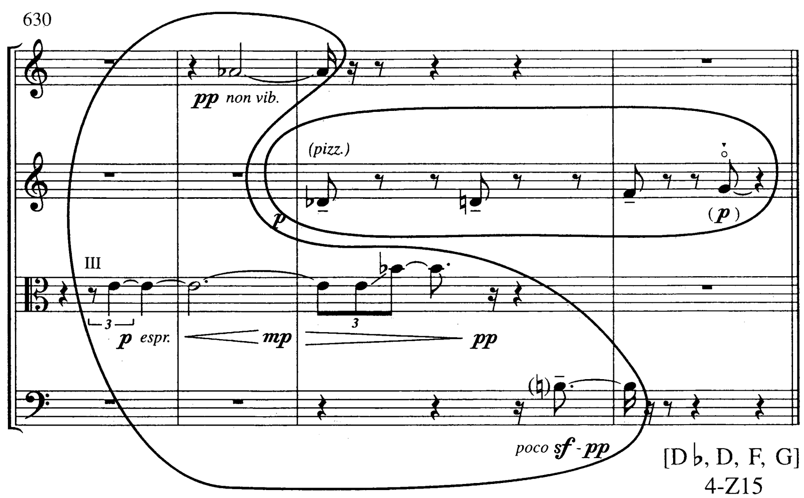
Z-Relation

Any two sets related by transposition or inversion must have the same interval-class content. The converse, however, is not true. There are several pairs of sets (one pair of tetrachords and octachords, three pairs of pentachords and septachords, and fifteen pairs of hexachords) that have the same interval-class content, but are *not* related to each other by either transposition or inversion and thus are *not* members of the same set class. Sets that have the same interval content but are not transpositions or inversions of each other are called *Z-related sets,* and the relationship between them is the *Z-relation.* (The Z stands for "zygotic," meaning "twinned.")

Sets in the Z-relation will sound similar because they have the same interval-class content, but they won't be as closely related to each other as sets that are members of the same set class. If the members of a set class are like siblings within a tightly knit nuclear family, then Z-related sets are like first cousins.

Composers have been particularly interested in the two "all-interval" tetrachords: 4–Z15 (0146) and 4–Z29 (0137). They are called all-interval tetrachords because, as suggested by their shared interval-class vector, 111111, both tetrachords contain one occurrence of each of the six interval classes. Example 3–7 contains two passages from Elliott Carter's String Quartet No. 2. In the first passage (Example 3–7A), the second violin plays two 3s (E–G and F–A♭) while the viola plays two 6s (C–F♯ and E♭–A)—in this quartet Carter often differentiates instruments in this way by assigning each a distinctive interval. The vertical combination of those intervals produces either 4–Z15 or 4–Z29. In the second passage (Example 3–7B), a form of 4–Z15 is stated melodically in the second violin while the other three instruments combine to create a form of 4–Z29.

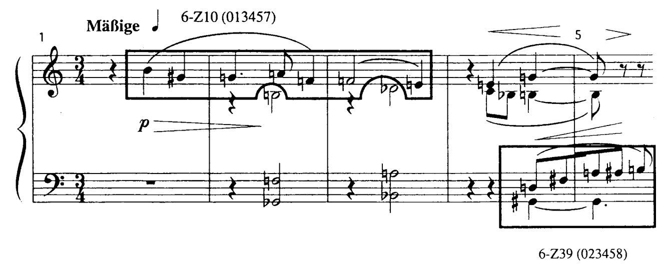




Example 3–7 The Z-relation (Carter, String Quartet No. 2).

In Example 3–8, the familiar beginning of the first of Schoenberg's Piano Pieces, Op. 11, the Z-relation creates a strong connection between the opening six-note melody and the left-hand accompanimental figure that follows.

Any set with a Z in its name has a *Z-correspondent,* another set with a different prime form but the same interval vector. On the set list in Appendix 1, the Z-related hexachords are listed across from one another, but you will have to look through the list for the Z-related sets of other sizes.



Example 3–8 The Z-relation (Schoenberg, Piano Piece, Op. 11, No. 1).

Complement Relation

For any set, the pitch classes it excludes constitute its *complement.* The complement of the set [3,6,7], for example, is [8,9,10,11,0,1,2,4,5]. Any set and its complement, taken together, will contain all twelve pitch classes. For any set containing n elements, its complement will contain 12 – n elements.

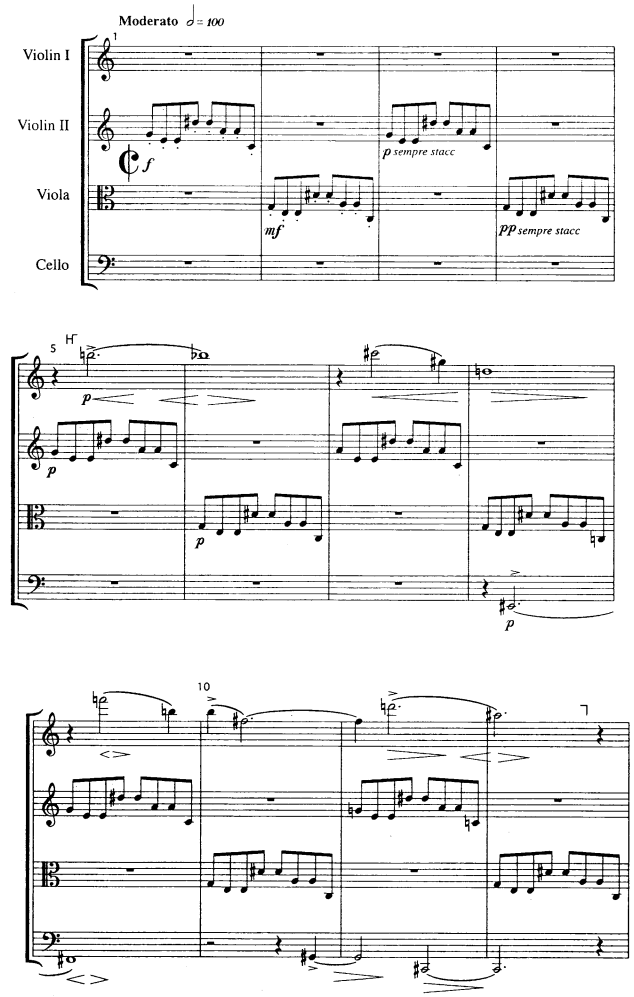
There is an important intervallic similarity between a set and its complement. You might think that whatever intervals a set has lots of, its complement will have few of, and vice versa. It turns out, however, that a set and its complement always have a proportional distribution of intervals. For complementary sets, the difference in the number of occurrences of each interval is equal to the difference between the size of the sets (except for the tritone, in which case the former will be half the latter). If a tetrachord has the interval-class vector 021030, its eight-note complement will have the vector 465472. The eight-note set has four more of everything (except for the tritone, of which it has two more). The larger set is like an expanded version of its smaller complement.

Because interval content is not changed by transposition or inversion, this intervallic relationship remains in force even when the sets are transposed or inverted. Thus, even if the sets are not *literally complementary* (i.e., one contains the notes excluded by the other), the intervallic relationship still holds so long as the sets are *abstractly complementary* (i.e., members of complement-related set classes). For example, [0,1,2] and [0,1,2,3,4,5,6,7,8] are not literal complements of each other. In fact, all the members of the first set are contained in the second. However, they are members of complement-related set classes and thus have a similar distribution of intervals. Complement-related sets do not have as much in common as transpositionally or inversionally related sets, but they do have a similar sound because of the similarity of their interval content.

The affinity of complement-related set classes extends beyond their intervallic content. Complement-related set classes always have the same degree of symmetry and thus the same number of sets in the class. If set X is Z-related to set Y, then the complement of X will be Z-related to the complement of Y. There are the same number of trichord- and nonachord-classes (12), of tetrachord- and octachord-classes

(29), and of pentachord- and septachord-classes (38) (hexachords will be discussed later). In each of these ways, sets and set classes resemble their complements.

The complement relation is particularly important in any music in which the twelve pitch classes are circulating relatively freely and in which the aggregate (a collection containing all twelve pitch classes) is an important structural unit. Consider the relatively common situation at the beginning of Schoenberg's String Quartet No. 3, where a melody (here divided between first violin and cello) is accompanied by an ostinato that contains all the pitch classes excluded by the melody (Example 3–9).



Example 3–9 Complementary sets in melody and accompaniment (Schoenberg, String Quartet No. 3).

The melody and the accompaniment have a similar sound because they contain a similar distribution of intervals.

The final four-note chord of the second of Schoenberg's Little Piano Pieces, Op. 19, is a form of 4–19 (0148), a set prominent throughout that piece and common in much of Schoenberg's music (Example 3–10).

Example 3–10 The complement relation (Schoenberg, Little Piano Piece, Op. 19, No. 2).

The last eight notes of the piece (which, of course, include that final four-note chord) are a form of 8–19 (01245689), the complementary set class. Compare the interval-class vectors of these two sets: the vector for 4–19 is 101310 and the vector for 8–19 is 545752. Both sets are particularly rich in interval class 4. In fact, no four- or eight-note set contains more 4s than these do. And notice how prominently the 4s are featured in the music. Because of the complement relation, the final four-note chord sounds similar to the larger eight-note collection of which it is a part.

The list of sets in Appendix 1 is arranged to make it easy to see the complement relation. Complementary set classes are always listed right across from one another. If you look up 4–19 (0148) and 8–19 (01245689), you will see that this is so. As a further aid, the names of complementary sets always have the same number following the dash. Thus, 4–19 and 8–19 are complements of each other, as are 3–6 and 9–6, 5-Z12 and 7-Z12, and so on. These features of the list make it very easy to look up large sets. Say you have a nine-note set that you want to find on the list. You could put it in prime-form and look it up, but that would be a time-consuming operation since the set is so big. It is far easier to take the three notes *excluded* by the nine-note set and put them in the prime-form, then look up that trichord on the list—the prime-form of the original nine-note set will be directly across from it.

You may notice that there are some sets, exclusively hexachords, that have nothing written across from them. Hexachords like that are *self-complementary*—they and their complements are members of the same set class. For a simple example, consider the hexachord [2,3,4,5,6,7]. Its complement is [8,9,10,11,0,1]. But both of these sets are members of set class 6–1 (012345). In other words, self-complementary hexachords are those that can map onto their complements under either T_n or T_nI.

If a hexachord is not self-complementary, then it must be Z-related to its complement. Remember that, with complementary sets, the difference in the number of occurrences of any interval is equal to the difference in the size of the two sets. But a hexachord is exactly the same size as its complement. As a result, a hexachord always has exactly the same interval content as its complement. If it is also related to its complement by T_n or T_nI, then it is self-complementary. If not, then it is Z-related to its complement. The hexachords on the list are thus either written with nothing across from them or they are written across from their Z-correspondents. This intervallic relationship between complementary hexachords is extremely important for twelve-tone music, and we will discuss it further in subsequent chapters.

Subset and Superset Relations

If set X is included in set Y, then X is a subset of Y and Y is a superset of X. A set of n elements will contain 2^n (2 to the nth power) subsets. A five-note set, for example, will contain the following subsets: the null set (a set containing no elements), five one-note sets, ten two-note sets (these are also called intervals), ten three-note sets, five four-note sets, and one five-note set (the original set itself). That makes a total of 2^5 (2 to the 5th power) or thirty-two subsets. The null set, the one-note sets, and the set itself will usually not be of particular interest as subsets. Even so, that still leaves lots of subsets to be considered ($2^n - (n + 2)$), and naturally, the bigger the set the more numerous the subsets.

In order not to be overwhelmed by the possibilities, there are two things to bear in mind. First, some of the subsets may be members of the same set class. Consider set class 4–25 (0268), for example, which is something of an extreme case. As Figure 3–11 shows, all of its three-note subsets are members of the same set class, 3–8 (026).

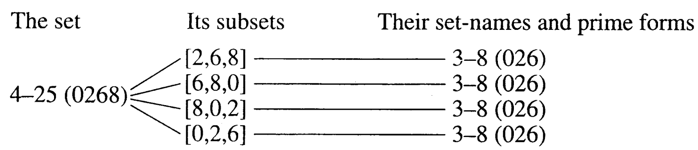


Figure 3–11

Most set classes are not as restricted in their subset content as this one, but there is often some redundancy.

To get a complete picture of the subset content of a set, it may be useful to construct an *inclusion lattice,* which lists all of the subsets of a given set as well as the subsets of those subsets. Figure 3–12 contains an inclusion lattice for set-class 6–20

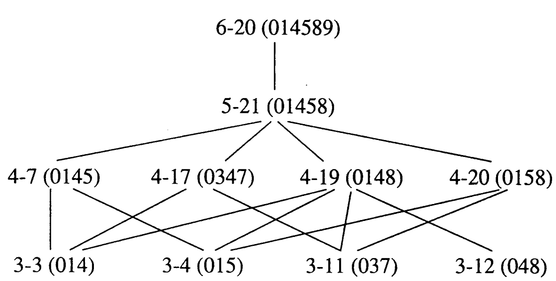
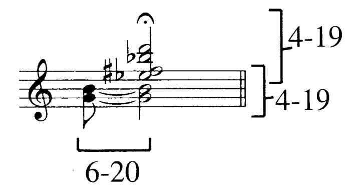


Figure 3–12

(014589), a set class also known as the hexatonic collection (to be discussed in Chapter 4). All six of the five-note subsets of 6–20 are members of set-class 5–21. The five four-note subsets of 5–21 in turn represent four different tetrachord classes, and these contain certain trichord classes as their subsets.

The final six-note chord of Schoenberg's Op. 19, No. 2 (discussed earlier with reference to Example 3–10), is a member of set-class 6–20—see Example 3–11. Schoenberg has arranged the chord so that its highest and lowest four notes represent set-class 4–19 and its highest and lowest three notes are augmented triads (set-class 3–12). Comparing the music in Example 3–11 with the inclusion lattice in Figure 3–12 gives a sense of what Schoenberg did in relation to what he might have done.



Example 3–11 Set-class 6-20 (014589) arranged to project two forms of set-class 4-19 (0148) as registral subsets (Schoenberg, *Little Piano Piece,* Op. 19, No. 2).

That brings us to the second important limitation on the otherwise vast world of subsets and supersets: only a small number will be musically significant in any specific musical context. Like any six-note set, the final sonority of Schoenberg's Little Piano Piece contains many subsets, but only a small number of those can be heard as meaningful musical groupings, identified by shared register or articulation. For example, it makes no musical sense to combine the G in the middle register with the top three notes—F♯, B♭, D—even though that combination creates another form of set class 4–19 (0148). Those four notes simply don't belong together musically. With the same final six-note sonority, Schoenberg could have grouped G, F♯, B♭, and D together, but chose not to. Similarly, he could have revoiced the sonority to emphasize subsets that were members of set classes other than 4–19, but again he chose not to. The subsets of a set are a kind of abstract musical potential; the composer chooses which to emphasize and which to repress.

As with the complement relation, the subset/superset relation can be either literal or abstract. Set X is a literal subset of Set Y if all of the notes of X are contained in Y. Set X is the abstract subset of Set Y if any transposed or inverted form of X is contained in Y, that is, if any member the set class that contains X is found among the subsets of Y. [E, F, G] is the literal subset of [C♯, D, E, F, G]. The T_5 transposition of [E, F, G], [A, B♭, C], is not a literal subset of [C♯, D, E, F, G]. But the set-class that contains it, 3–2 (013), can be found among the literal subsets of [C♯, D, E, F, G]—both [C♯, D, E] and [E, F, G] represent it. So [A, B♭, C] is an abstract, not a literal, subset of [C♯, D, E, F, G]. In the same abstract sense, we would say that set-class 3–2 is a subset of set-class 5–10.

In both the literal and abstract senses, these "inclusion" relations are not as strong as many of the relationships discussed earlier, like the Z-relation or the complement relation, but can still be musically interesting. Smaller collections can frequently be heard combining into larger ones and larger collections dividing into smaller ones.

Transpositional Combination

The process of combining smaller sets to form larger ones and dividing larger sets into smaller ones is particularly interesting when the smaller sets are related by inversion or transposition. We have already discussed inversional symmetry. Any time you combine two sets related by inversion, you get a set that is inversionally symmetrical. Conversely, any inversionally symmetrical set can be divided into at least one pair of inversionally related subsets. *Transpositional combination* (TC) is the combination of a set with one or more transpositions of itself to create a larger set. The larger set, which can then be divided into two or more subsets related by transposition, is said to have the TC property, and sets with this property have often proven of interest to composers.

In Example 3–12, from Stravinsky's *Symphony of Psalms,* the bass part (cellos and contrabasses) begins with ip3, F–A♭.

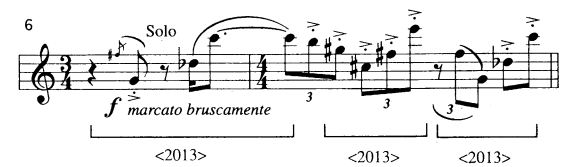
Example 3–12 Transpositional combination (Stravinsky, *Symphony of Psalms,* first movement).

Another ip3, E–G, follows immediately a semitone lower. That combination of two 3s a semitone apart is written 3*1, where the asterisk stands for "transposed by." One also could think of the figure as two semitones (E–F and G–A♭) related at T_3, or 1*3. Either way, that combination of 1 and 3 produces a form of sc4–3 (0134). The same combination produces a different member of the same set class, [B♭, B, C♯, D] in the alto voice (oboes and english horn). These two tetrachords, created by transpositional combination, are themselves combined at T_6 to create an eight-note set. We can summarize the process as: (3*1) * 6. In other words, a 3 is transposed by 1, and the resulting tetrachord is transposed at T_6. The passage can thus be built up from its smallest components through transpositional combination.

Contour Relations

Throughout this book thus far we have focused on pitches, pitch classes, and their intervals. We have explored ways that lines and sets of pitches and pitch classes can move through and be related in pitch-space and pitch-class space. And the relationships have been, in some cases, quite abstract. As listeners, we may sometimes find it easier to attend to the general shapes of music, its motions up and down, higher and lower. These are aspects of musical *contour.* To make sense of musical contour, we do not need to know the exact notes and intervals; we only need to know which notes are higher and which are lower.

Compare three four-note fragments from a melody from Crawford's String Quartet (see Example 3–13).



Example 3–13 A recurring contour-segment (CSEG) (Crawford, String Quartet, first movement, mm. 6–7).

The fragments are intervallically distinct, and represent three different set classes. But their contours are the same. Each begins on its second-highest note, continues with its lowest and its second-lowest notes, and concludes on its highest note. In Example 3–13, that contour is represented as a string of numbers enclosed within angle-brackets: <2013>. The notes in each fragment are assigned a number based on their relative position in the fragment. 0 is assigned to the lowest note, 1 to the next-lowest, and so on. The highest note will always have a numerical value that is 1 less than the number of different notes in the fragment. The numbers are then arranged, in order, to describe the musical contour. <2013> is a *contour segment,* or *CSEG,* and this intervallically varied melody is unified, in part, by three presentations of that single CSEG.

At the end of the movement, the second violin has a varied version of the same melody (see Example 3–14).

Example 3–14 Members of a CSEG-class (Crawford, String Quartet, first movement, mm. 72–75).

The notes are different, but the same CSEG, <2013> is represented three times (Example 3–14a). The CSEG created by the four notes beginning on D is <1320> (Example 3–14b). <2013> and <1320> are related by inversion. The highest note in one is replaced by the lowest note in the other, the second-highest by the second-lowest, and so on. They are mirror images of each other. And just as when we compare two lines of pitch classes, the numbers in the corresponding order positions always add up to the same sum, in this case 3. One additional CSEG, <0231>, occurs twice (Example 3–14c). <0231> and <1320> are retrograde-related—each is the same as the other written backwards. Similarly, <0231> and <2013> are related by retrograde-inversion—each is the inverted and backwards version of the other.

Like pitch-class sets, CSEGs can be gathered into CSEG-classes. CSEGs related by inversion, retrograde, or retrograde-inversion belong to the same CSEG-class. The three CSEGs of Example 3–14, and one more that is not shown, <3102>, are the four members of a single CSEG-class. Crawford's violin melody seems to be interested in the reshaping of this basic shape. Of the four members of this CSEG-class, we select the one that begins on the lowest note to act as prime form. Crawford's melodic fragments all belong to the CSEG-class with prime-form <0231>.

The CSEG-classes for CSEGs of three and four notes are listed in Figure 3–13 (the CSEG-classes proliferate rapidly after that). Approaching contour in this way permits us to discuss musical shapes and gestures with clarity, but without having to rely on more difficult discriminations of pitches, pitch-classes, and their intervals. Contour can be particularly revealing, however, when studied in relationship to pitch and pitch class. There, it becomes possible to discuss similarities of shape in the presentation of different set classes and, conversely, the divergent shapes given to members of the same set class.

Contour can also be useful in talking about musical elements other than pitch. In Example 3–15, a measure from a piano piece by Stockhausen, the right-hand melody (D7–C♮6–C5–G5) and its dynamics (fff–ff–pp–p) can both be understood as expressing CSEG <3201>. Just as the pitches move from highest to second-highest

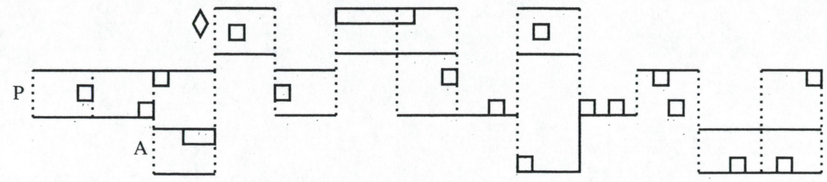
Name	*Prime-form*
3–1	<012>
3–2	<021>
4–1	<0123>
4–2	<0132>
4–3	<0213>
4–4	<0231>
4–5	<0312>
4–6	<0321>
4–7	<1032>
4–8	<1302>

Figure 3–13

to lowest to second-lowest, the dynamics move from loudest to second-loudest to softest to second-softest. The pitch contour and the dynamic contour are the same.

Example 3–15 Same contour, CSEG <3201>, expressed in both pitch and dynamics (Stockhausen, *Klavierstück* II, m. 14).

Contour can be particularly valuable as a way of talking about music that is indefinite as to pitch, like a lot of experimental music written since 1950. Example 3–16 shows the opening of Morton Feldman's *Projection No. 1 for Solo Cello.* The dotted lines function as barlines, and each measure contains four beats (at a tempo of beat = 72). Each square or rectangle indicates a musical event, and the duration of the event is shown by the length of the rectangles, which last for between one and five beats in this passage. There are three different timbres indicated (diamond = harmonics; P = pizzicato; A = arco) and, within each timbre, relative pitch is indicated by the vertical position of the square or rectangle. The first three pizzicato notes, for example, fall in a middle register, a low register, and then a high register—they thus describe CSEG <102>. Now consider the temporal distance between the pizzicato notes: four beats from the first note to the second; one beat from the second to the third; and eight beats from the third to the fourth. That durational contour also can be described by CSEG <102>: a medium duration followed by a short duration and then a long duration. Similar kinds of patterning can be found elsewhere in the piece, structuring the pitches and the durations both independently and in relation to each other.

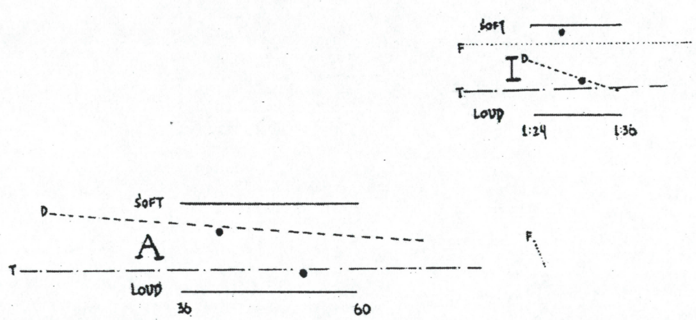


Example 3–16 Same contour, CSEG <102> expressed by both pitch and duration (Feldman, *Projection No. 1 for Solo Cello*).

A complete piece by John Cage is shown in Example 3–17. The piece consists of four events, indicated by dots, and these occur at the times indicated below the graphics—the first two events occur between thirty-six and sixty seconds after the beginning of the piece and the third and fourth events occur between 1:24 and 1:36. The first two events are to be played in some way that does not involve the piano (that is the meaning of the letter A, for "auxiliary sounds") while the last two events are to be played in the interior of a piano (I stands for "interior"). Beyond that, each of the four events is characterized by four qualities: (1) duration, measured by proximity to the dotted line labeled D; (2) dynamics, measured by position between the solid lines marked soft and loud; (3) pitch, measured by distance from the dotted line labeled F, which indicates the lowest frequencies; and (4) timbre, measured by distance from the line marked T, which indicates the fewest possible overtones. Within each of those four domains, the four events of the piece describe a particular contour:

Durations: <1230> Dynamics: <1302> Pitch: <3201> Timbre <2031>

All are different, which may suggest the structural heterogeneity of this work, its resistance to analytical coherence. By contrast, the dynamic and timbre contours are inversions of each other, that is, the louder the note the least rich in overtones, and the richer in overtones the softer. That may suggest some degree of coordination among the dimensions. Either way, contour provides a useful vantage point for apprehending the piece.



Example 3–17 Contours of duration, dynamics, pitch, and timbre (Cage, *For Paul Taylor and Anita Dencks*).

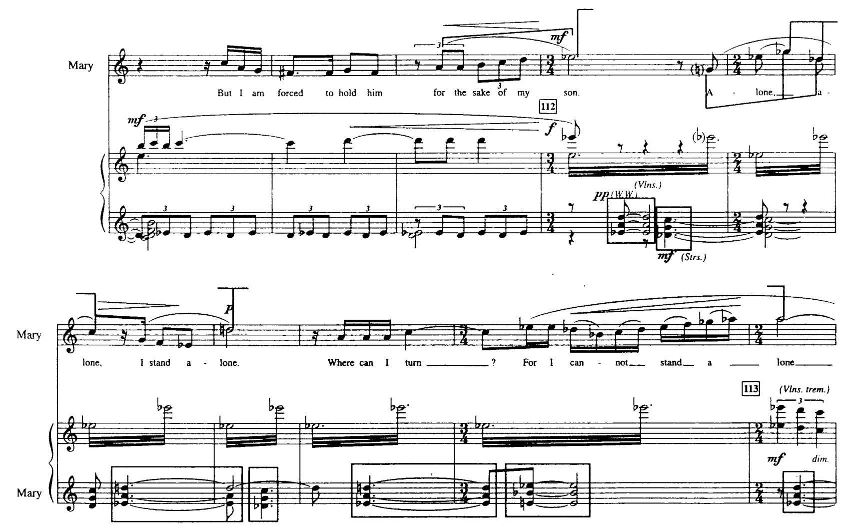
Composing-Out

To organize the larger musical spans and draw together notes that may be separated in time, composers of post-tonal music sometimes enlarge the motives of the musical surface and project them over significant musical distances. This musical procedure is sometimes called *composing-out*—we would say that a motive from the musical surface is *composed-out* at a deeper level of structure (other related terms are "enlargement," "concealed repetition," "motivic parallelism," "nesting," and "self-similarity").

Example 3–18 shows the first four vocal phrases of a song by Webern. The melody begins with four notes: D–D♭–E♭–G♭. The same notes, in the same order, are composed-out as the first notes of the four vocal phrases (Example 3–18A). A somewhat more subtle kind of composing-out involves the last four notes of the first phrase: F–A♭–E–B♭ (Example 3–18B). These return, reordered and transposed at T_6, in the last notes of the four vocal phrases. The boundary notes at the beginning and ends of phrases thus compose-out the notes and intervals of the musical surface.

Example 3–18 Composing-out (Webern, Song, Op. 3, No. 1).

In the aria from Thea Musgrave's *Mary Queen of Scots* shown in Example 3–19, the accompaniment consists of a single chord, E♭–A–D, which is transposed downward and then back up to its starting point. The same three notes, although in different order and registral arrangement, are also stated as the long notes within the vocal line. Two additional statements of the same set class—3–5 (016)—may be found within the vocal line, and these are indicated in Example 3–19. In this way, the notes of a harmony are composed-out melodically over the span of an entire phrase.



Example 3–19 Composing-out (Musgrave, *Mary, Queen of Scots,* excerpt from Act III).

Example 3–20 reproduces the first section of the first of Schoenberg's Piano Pieces, Op. 11, a passage we have already looked at several times. We noted previously the extent to which set class 3–3 (014) pervades the musical surface. Example 3–20 shows two large-scale statements of the same set class, one in the upper voice and one in the bass. As we observed in Chapter 2, the three melodic high points, B–G–G♯, constitute a large-scale statement of set class 3–3. These three pitches, widely separated in time, are associated by their shared register and contour position. Furthermore, these are the same three pitches with which the piece began.

A similar thing happens in the bass. The left-hand part begins with two chords (measures 2–3); the bass notes are G♭ and B♭. After contrasting material, two more chords are heard at the end of the section (measures 10–11); the bass notes now are G♭ and G. That final G completes a large-scale statement of another form of set class 3–3. These three pitches, G♭–B♭–G, are associated by their shared register and articu-

Example 3–20 Large-scale statements, in soprano and bass, of set class 3–3 (014) (Schoenberg, Piano Piece, Op. 11, No. 1).

lation. Like the large-scale melodic statement, this large-scale bass statement draws together and unifies this section of music.

Linear projections of this kind may extend over very large spans of music, including entire pieces. Stravinsky's ballet *Les Noces (The Wedding)* begins with the melody in Example 3–21a, which consists of [B, D, E] (the grace note F♯ is excluded). At the beginning of the third scene of the ballet, that set is transposed two semitones higher and repeated with extraordinary insistence (Example 3–21b). The ballet concludes with a protracted coda that consists of slow, obsessive repetitions of still another transposition of the original fragment, now five semitones below the original: [G♯, B, C♯] (Example 3–21c). The large-scale progression, one that spans the entire ballet, thus composes-out the intervallic shape of the original motive (Example 3–21d). Here is composing-out over a truly monumental span!

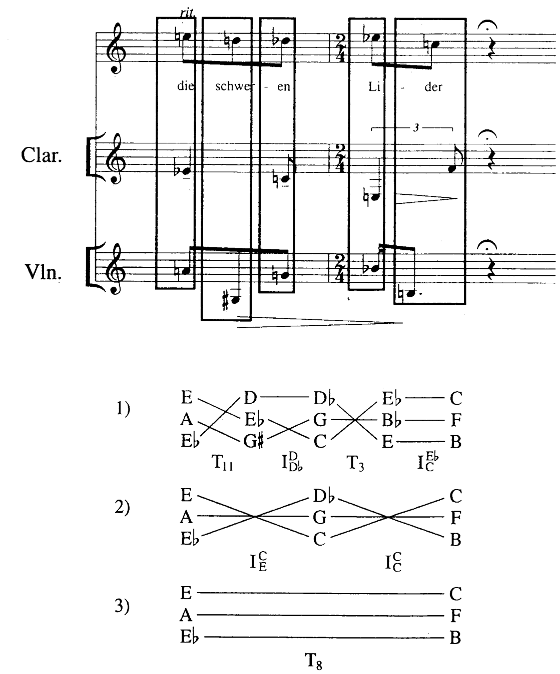
Example 3–21 Composing-out (Stravinsky, *Les Noces*).

Voice Leading

* pc sets that belong to the same set class

One useful way of describing the voice leading of post-tonal music involves attending to the pitch-class counterpoint created by transposition and inversion. As we have seen, transposition and inversion involve mapping notes from one set to the next. Those mappings can be understood to comprise post-tonal voices, which move through the musical texture.

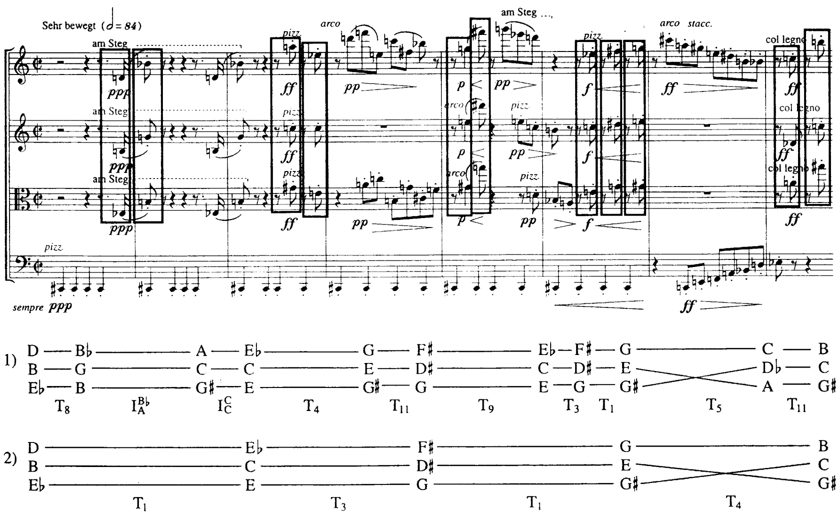
In Example 3–22, from a song by Webern, the boxed chords are all members of set-class 3–5 (016).



Example 3–22 Transformational voice leading (Webern, Songs, Op. 14, "Die Sonne," mm. 23–24).

The horizontal and diagonal lines trace the pitch-class mappings induced by the specified operations. Three voices move through the progression. One begins on E and moves down to the bottom of the third chord before returning to its original position in the highest register. The middle and lowest voices also move through the chords and return to their original position at the end. The second level of the analysis simplifies the five-chord progression into two inversional moves, each of which exchanges the vocal part with the lowest sounding part. Finally, the third level describes the progression as the transposition at T_8 (actually a pitch transposition down four semitones) of the first chord onto the last chord. At each level, the harmonies are bound together by the motions of the voices.

The passage in Example 3–23, from Webern's Movements for String Quartet, Op. 5, No. 3, consists of groups of two or three chords, all members of set-class 3–3 (014), interspersed with canonic interjections.

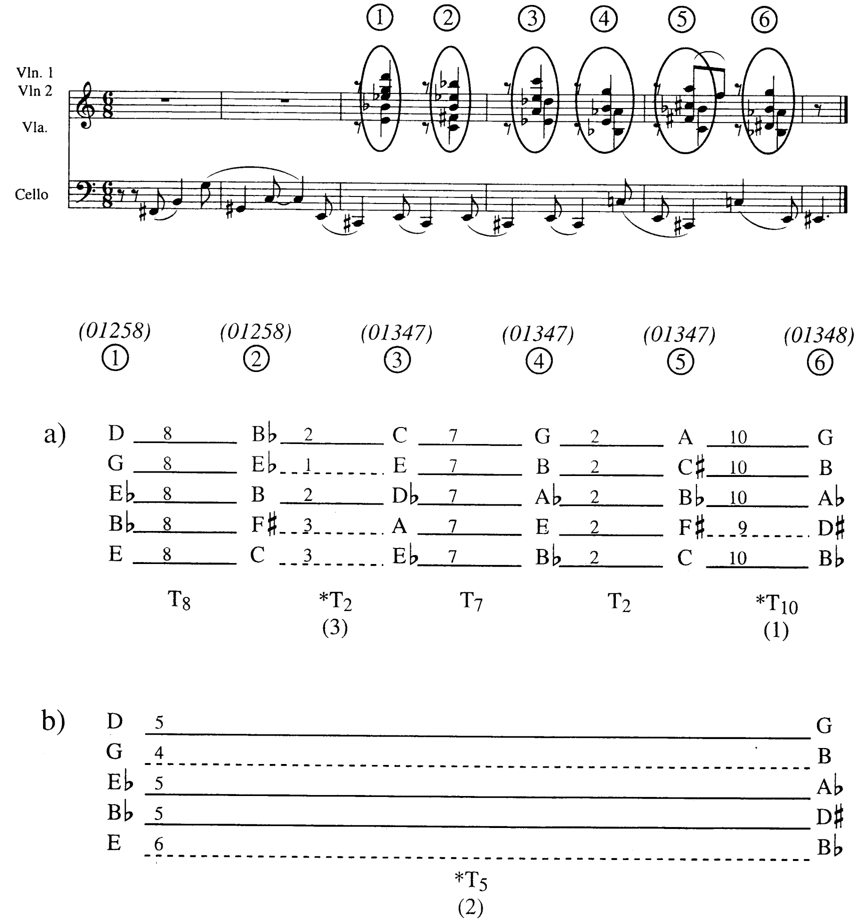


Example 3–23 Transformational voice leading (Webern, Movements for String Quartet, Op. 5, No. 3, mm. 1–8).

The instrumentation and registration of the chords reflects their voice leading to a remarkable degree, exclusively so in the first violin, and one with one brief voice crossing in the second violin and viola.

The second level of the analysis isolates one chord from each group of chords: the first chord from the first group and the last chord in each group thereafter. The intervals of transposition—T_1, T_3, and T_4—are the same as the intervals contained within set-class 3–3 (014), which is the chord being transposed. The voice leading thus follows a motivic path.

Another movement from the same piece also involves a progression of chords in the upper three instruments (see Example 3–24). These six chords represent three different set classes: the first and second chords are related by transposition, as are the third, fourth, and fifth chords. The analytical problem is to connect the second chord to the third and the fifth chord to the sixth. In a situation like this, it may be useful to imagine that the chords are almost-but-not-quite related by transposition (or inversion). In the progression from the second chord to the third, we can describe the relationship as a quasi-T_2 or a *fuzzy*-T_2: three of the voices deviate from T_2, but they do so as little as possible, by only a semitone each (Example 3–24a). In moving from the fifth chord to the sixth, the deviation from an actual transposition is even smaller—only one voice is off, and it's off by only one semitone. Transpositions (or

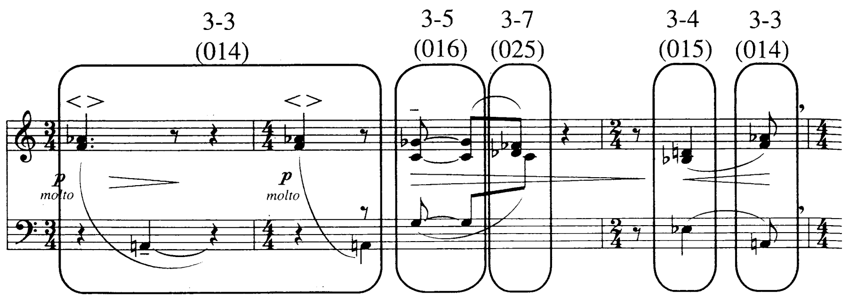


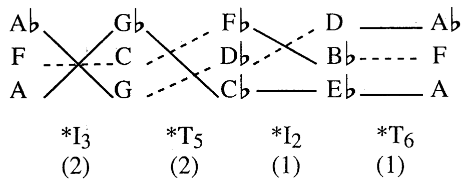
Example 3–24 Transformational voice leading, with fuzzy transposition (Webern, Movements for String Quartet, Op. 5, No. 5).

inversions) that are a bit off are designated with an asterisk and the amount of the deviation, known as the *offset,* is indicated in parentheses. In the voice leading, actual mappings are shown with a solid line, fuzzy mappings with a dotted line.

In this passage, the individual transpositional moves combine to create a single, larger transpositional move: from the first chord to the last is a fuzzy-transposition at $*T_5$, with an extremely small offset of only two semitones (Example 3–24b). The progression as a whole, embracing members of three different set classes, can thus be heard as a single, unified gesture, with clear, parallel voice leading.

Of course, the voice leading does not have to be parallel. In the passage shown in Example 3–25, there are five chords representing four different set classes. The fuzzy transpositions and inversions that connect the chords produce a voice leading in which the voices cross.





Example 3–25 Transformational voice leading, with fuzzy transposition and inversion (Sessions, Piano Sonata, first movement).

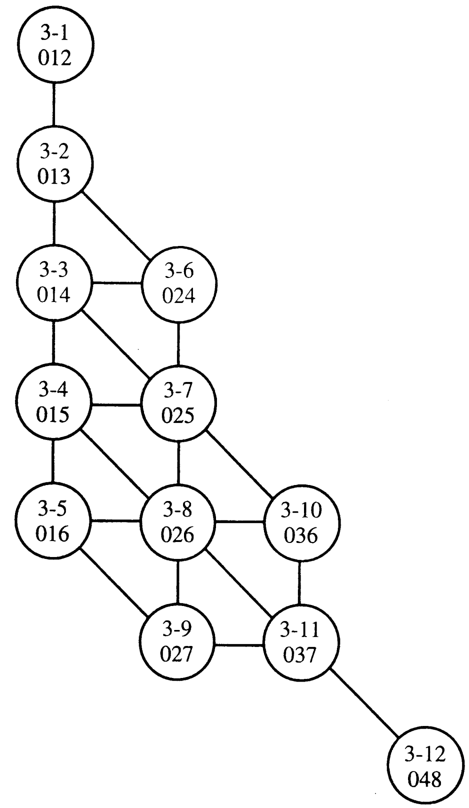
Atonal Pitch Space

In discussing voice leading, we were talking about actual sets of pitch classes and the ways in which individual notes in one set move onto individual notes in another. It is also possible to talk more abstractly about the voice leading between and among set classes. Given a set belonging to one set class, we might ask how much semitonal adjustment would be necessary to turn it into a member of a different set class.

As an example, take the trichord [G, A, B], a member of sc(024). Let's see what happens if we adjust each of its notes, in turn, either up or down by semitone. We would get six different trichords representing three different set classes:

[G♯, A, B] = (013)
[F♯, A, B] = (025)
[G, A♭, B] = (014)
[G, B♭, B] = (014)
[G, A, B♭] = (013)
[G, A, C] = (025)

In general terms, we can say that sc(024) is offset by a distance of only one voice-leading semitone from sc(013), sc(014), and sc(025). If we applied the same procedure to all of the trichord-classes, we would end up with a map like the one in Example 3–26.



Example 3–26 Voice-leading space for trichords.

In this *voice-leading space,* the trichords are shown in relative proximity to each other. The closer two trichords are on the map, the less semitonal distance there is between them. Each line on the map represents one semitone of offset (that is, one semitone of voice-leading adjustment or voice-leading distance), and the distances accumulate in a consistent way. From (012), one semitone of offset will take you to (013), two semitones of offset will take you to (014) or (024), three semitones of offset will take you to (015) or (025), and so on, up to the maximum of six semitones of offset between (012) and (048).

As the size of the sets increases, it becomes harder to represent the space on a two-dimensional page. But the trichordal map in Example 3–26 suggests the possibility of understanding a passage of music as a journey through a voice leading space defined by semitonal offset—sometimes the moves will be to nearby destinations, involving only a small amount of adjustment; other times the music may make large harmonic leaps.

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